JOURNAL OF APPROXIMATION THEORY **50**, 185–191 (1987)

Equiconvergence of Some Lacunary Trigonometric Interpolation Polynomials

A. K. VARMA

Department of Mathematics, University of Florida, Gainesville, Florida 32611, U.S.A.

AND

P. VÉRTESI

Mathematical Institute, Hungarian Academy of Sciences, Reàltanoda u. 13–15 Budapest H-1053, Hungary

Communicated by Paul G. Nevai

Received January 13, 1984; revised November 20, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION

We study in this paper the problem of equiconvergence of some lacunary (Birkhoff) trigonometric interpolation polynomials (for a detailed treatment of this subject we refer to the recent book of Lorentz *et al.* [1]). In 1968 A. Sharma and A. K. Varma [6] proved that there exists a unique trigonometric polynomial $R_n(x)$ such that

$$R_n(x_{kn}) = a_{kn}, \qquad R_n''(x_{kn}) = b_{kn}, \qquad R_n'''(x_{kn}) = c_{kn}, \tag{1.1}$$

where $x_{kn} = 2\pi k/n$ and a_{kn}, b_{kn}, c_{kn} are given constant, $k = 0, 1, ..., n-1$ ("(0, 2, 3) interpolation"). We require the trigonometric polynomial to have the form

$$d_0 + \sum_{k=1}^{M-1} (d_k \cos kx + e_k \sin kx) + \varepsilon d_M \cos Mx, \tag{1.2}$$

where $M = [(3n+1)/2]$ and $\varepsilon = 1$ or 0 according as n is even or odd.

In the same paper [6] the following convergence theorem was proved:

0021-9045/87 $3.00

THEOREM A (A. Sharma and A. K. Varma [6]). *Let f be 2π-periodic and in* Lip α, $\alpha > 0$, *and let*

$$a_{kn} = f(x_{kn}),\ b_{kn} = o\left(\frac{n^2}{\log n}\right), \quad c_{kn} = o\left(\frac{n^3}{\log n}\right), \quad k = 0, 1, ..., n-1. \tag{1.3}$$

Then $R_n(x) \equiv R_n(f, x)$ satisfying (1.1) *and* (1.3) *converges uniformly to $f(x)$ on the whole real line.*

It is natural to ask to what extent is this theorem best possible? This problem was examined in [8]. It was shown that even in the case $b_{kn} = c_{kn} = 0$, $f \in \text{Lip}\ \alpha$, $\alpha > 0$, cannot be replaced by $f \in C_{2\pi}$.

Throughout this paper we shall assume $n = 2m + 1$,

$$x_{kn} = 2\pi k/n, \qquad k = 0, 1, ..., n-1. \tag{1.4}$$

Further, we shall denote by $Q_n(f, x)$ the trigonometric polynomial of the form (1.2) satisfying

$$Q_n(f, x_{kn}) = f(x_{kn}), \qquad Q_n''(f, x_{kn}) = Q_n'''(f, x_{kn}) = 0, \\ k = 0, 1, ..., n-1. \tag{1.5}$$

It is well known [see Zygmund [10]] that the unique trigonometric polynomial of order m which interpolates $f(x)$ at the nodes (1.4) is given by

$$T_m(f; x) = \frac{1}{n} \sum_{k=0}^{n-1} f(x_{kn})\, D_m(x - x_{kn}) \tag{1.6}$$

with

$$D_m(x) = 1 + 2 \sum_{i=1}^{m} \cos ix. \tag{1.7}$$

It would be nice to observe that $T_m(f; x)$ converges to $f(x)$ only under certain conditions and this would motivate equiconvergence of $f - Q$ and $f - T_m$. For interesting contributions on equiconvergence we refer to G. Szegö [7] (see specifically Chapters IX and XIII). More precisely, we state our main theorem on equiconvergence as follows:

THEOREM 1. *Let $f \in C_{2\pi}$, then there exists a function $\mu_n(x) \in c_{2\pi}$ independent of f with $|\mu_n(x)| < c$ such that*

$$\lim_{n \to \infty} [f(x) - Q_n(f, x) - \mu_n(x)(f(x) - T_m(f, x))] = 0, \tag{1.8}$$

uniformly on $0 \leqslant x \leqslant 2\pi$. *Here* $Q_n(f, x)$ *and* $T_m(f, x)$ *are defined by* (1.5) *and* (1.6).

COROLLARY 1. *Let* $p > 0$ *be any fixed number. Then the sequence* $Q_n(f, x)$ *defined by* (1.7) *satisfies*

$$\lim_{n \to \infty} \int_0^{2\pi} |Q_n(f, x) - f(x)|^p \, dx = 0$$

for every $f \in C_{2\pi}$.

The above corollary at once follows from Theorem 1 and the well-known theorem of Marcinkiewicz [2, 10] which is stated below.

THEOREM B. *Let* $f(x) \in c_{2\pi}$, $0 < p < \infty$, *and* $T_m(f, x)$ *is given by* (1.6) *then*

$$\lim_{n \to \infty} \int_0^{2\pi} |f(x) - T_m(f, x)|^p \, dx = 0. \tag{1.9}$$

For many interesting contributions on the problem of mean convergence we refer to the works of P. Nevai [3–5], P. Vértesi [9], and many references mentioned therein.

2. PRELIMINARIES

We denote the Fejér kernel by

$$t_{1,k}(x) = 1, \qquad t_{j,k}(x) = 1 + \frac{2}{j} \sum_{i=1}^{j-1} (j-i) \cos i(x - x_{kn}), \qquad j > 1. \tag{2.1}$$

The following properties of the Fejér kernel will be needed:

$$(j+1)\, t_{j+1,k}(x) - 2j t_{j,k}(x) + (j-1)\, t_{j-1,k}(x) = 2 \cos j(x - x_{kn}), \tag{2.2}$$

$$\sum_{k=0}^{n-1} t_{j,k}(x) = n, \tag{2.3}$$

$$t_{j,k}(x) = \frac{1}{j} \left(\frac{\sin j \dfrac{x - x_{kn}}{2}}{\sin \dfrac{x - x_{kn}}{2}} \right)^2. \tag{2.4}$$

From (1.10) we have

$$D_m(x-x_{kn}) = 1 + 2\sum_{i=1}^{m} \cos i(x-x_{kn}), \tag{2.5}$$

$$D_m(x-x_{kn}) = (m+1)\, t_{m+1,k}(x) - m t_{m,k}(x), \tag{2.6}$$

$$D_{3m+1}(x-x_{kn}) = (3m+2)\, t_{3m+2,k}(x) - (3m+1)\, t_{3m+1,k}(x), \tag{2.7}$$

$$D_{3m+1}(x-x_{kn}) = (1 + 2\cos(2m+1)\,x)\, D_m(x-x_{kn}). \tag{2.8}$$

In the quoted work [6] the explicit form of $Q_n(f, x)$ was also obtained. It is given by

$$Q_n(f, x) = \sum_{k=0}^{n-1} f(x_{kn})\, u(x-x_{kn}) \tag{2.9}$$

where for n odd ($= 2m+1$)

$$u(x) = A_n(x) - B_n(x) \tag{2.10}$$

where

$$A_n(x) = \frac{1}{n}\left[1 + \frac{2}{n^2}\sum_{j=1}^{m} \alpha_j \cos jx\right], \qquad B_n(x) = \frac{1}{n^3}\sum_{j=m+1}^{3m+1} \beta_j \cos jx \tag{2.11}$$

and

$$\alpha_j = \alpha_{j,n} = \frac{(n^2-j^2)^2}{n^2-3j^2}, \qquad \beta_j = \beta_{j,n} = \frac{(n-j)^2(2n-j)^2}{n^2-3(n-j)^2}. \tag{2.12}$$

For the proof of Theorem 1, we need to express the fundamental functions $u(x-x_{kn})$ in terms of Fejér and Dirichlet kernels. This result is shown in the next lemma.

LEMMA. *The following representation of $A_n(x)$ and $B_n(x)$ is valid*:

$$A_n(x-x_{kn}) = \frac{1}{n^3}\sum_{l=1}^{m-1} \gamma_l\, l t_{l,k}(x) + \frac{m}{n^3}(\alpha_{m-1} - \alpha_m)\, t_{m,k}(x) + \frac{\alpha_m}{n^3} D_m(x-x_{kn}) \tag{2.13}$$

and

$$B_n(x-x_{kn}) = \frac{1}{2n^3}\Bigg[\sum_{l=m+1}^{3m} \delta_l\, l t_{l,k}(x) + (\beta_{m+1} - \beta_m)\, m t_{m,k}(x) + (3m+1)(\beta_{3m} - \beta_{3m+1})\, t_{3m+1,k}(x) + (\beta_{3m+1}(1 + 2\cos nx) - \beta_m)\, D_m(x-x_{kn})\Bigg] \tag{2.14}$$

where

$$\gamma_l = \alpha_{l-1} - 2\alpha_l + \alpha_{l+1}, \qquad \delta_l = \beta_{l-1} - 2\beta_l + \beta_{l+1}. \tag{2.15}$$

Proofs of (2.13) and (2.14) are on the same lines. Therefore we give the details of (2.13) only. From (2.11), (2.2), and (2.15) we obtain

$$\begin{aligned} A(x - x_{kn}) &= \frac{1}{n}\left[1 + \frac{1}{n^2}\sum_{j=1}^{m} \alpha_j((j+1)\, t_{j+1,k}(x) - 2jt_{j,k}(x)\right. \\ &\quad \left. + (j-1)\, t_{j-1,k}(x))\right] = \frac{1}{n}\left[1 + \frac{1}{n^2}\left(\sum_{l=2}^{m+1} \alpha_{l-1} l t_{l,k}(x)\right.\right. \\ &\quad \left.\left. - 2\sum_{l=1}^{m} l\alpha_l t_{l,k}(x) + \sum_{l=1}^{m-1} \alpha_{l+1} l t_{l,k}(x)\right)\right] \\ &= \frac{1}{n^3}\left[\sum_{l=1}^{m-1} (\alpha_{l-1} - 2\alpha_l + \alpha_{l+1})\, l t_{l,k}(x)\right. \\ &\quad \left. + \alpha_{m-1}\, m t_{m,k}(x) + \alpha_m(m+1)\, t_{m+1,k}(x) - 2m\alpha_m t_{m,k}(x)\right] \\ &= \frac{1}{n^3}\left[\sum_{l=1}^{m-1} \gamma_l l t_{l,k}(x) + (\alpha_{m-1} - \alpha_m)\, m t_{m,k}(x)\right. \\ &\quad \left. + \alpha_m((m+1)\, t_{m+1,k}(x) - m t_{m,k}(x)\right] \\ &= \frac{1}{n^3}\left[\sum_{l=1}^{m-1} \gamma_l l t_{l,k}(x) + (\alpha_{m-1} - \alpha_m)\, m t_{m,k}(x) + \alpha_m D_{m,k}(x)\right]. \end{aligned}$$

This proves (2.13).

3. Proof of Theorem 1

Since $Q(1; x) \equiv 1$, we have

$$f(x) - Q_n(f, x) = \sum_{k=0}^{n-1} (f(x) - f(x_{kn}))\, u(x - x_{kn}).$$

Further, from (2.10), (2.11), and (2.13) we have

$$\begin{aligned} f(x) - Q_n(f, x) &= a_m(x) \sum_{k=0}^{n-1} (f(x) - f(x_{kn}))\, D_m(x - x_{kn}) \\ &\quad + b_m \sum_{k=0}^{n-1} (f(x) - f(x_{kn}))\, t_{m,k}(x) \end{aligned}$$

$$-\frac{(3m+1)}{2n^3}(\beta_{3m}-\beta_{3m+1})\sum_{k=0}^{n-1}(f(x)-f(x_{kn}))\,t_{3m+1,k}(x)$$
$$+\frac{1}{n^3}\sum_{l=1}^{m-1}\gamma_l l\sum_{k=0}^{n-1}(f(x)-f(x_{kn}))\,t_{l,k}(x)$$
$$-\frac{1}{2n^3}\sum_{l=m+1}^{3m}l\delta_l\sum_{k=0}^{n-1}(f(x)-f(x_{kn}))\,t_{l,k}(x), \tag{3.1}$$

where

$$a_m(x)=\frac{\alpha_m}{n^3}-\frac{1}{2n^3}(\beta_{3m+1}(1+2\cos nx)-\beta_m) \tag{3.2}$$

$$b_m=\frac{m}{n^3}(\alpha_{m+1}-\alpha_m)-\frac{m}{2n^3}(\beta_{m+1}-\beta_m). \tag{3.3}$$

From (3.2), (3.3) we have

$$a_m(x)=\frac{\mu_n(x)}{n}, \qquad |\mu_n(x)|\leqslant c_1, \tag{3.4}$$

$$|b_m|\leqslant\frac{c_2}{n}, \qquad |\beta_{3m}-\beta_{3m+1}|\leqslant c_3 n, \tag{3.5}$$

$$|\gamma_l|\leqslant c_4, \qquad l=1,2,\ldots,m-1, \tag{3.6}$$

$$|\delta_l|\leqslant c_5, \qquad l=m+1,\ldots,3m, \tag{3.7}$$

where c_1, c_2, c_3, c_4, and c_5 are independent of n. Therefore, we have

$$f(x)-Q_n(f,x)-\mu_n(x)(f(x)-T_m(f,x))$$
$$=\lambda_{2n}(x)+\lambda_{3n}(x)+\lambda_{4n}(x)+\lambda_{5n}(x), \tag{3.8}$$

where

$$\lambda_{2n}(x)=b_m\sum_{k=0}^{n-1}(f(x)-f(x_k))\,t_{m,k}(x),$$
$$\lambda_{3n}(x)=\frac{(3m+1)}{2n^3}(\beta_{3m+1}-\beta_{3m})\sum_{k=0}^{n-1}(f(x)-f(x_{kn}))\,t_{3m+1,k}(x),$$
$$\lambda_{4n}(x)=\frac{1}{n^3}\sum_{l=1}^{m-1}\gamma_l l\sum_{k=0}^{n-1}(f(x)-f(x_{kn}))\,t_{l,k}(x),$$

and

$$\lambda_{5n}(x) = \frac{-1}{2n^3} \sum_{l=m+1}^{3m} l\delta_l \sum_{k=0}^{n-1} (f(x) - f(x_{kn}))\, t_{l,k}(x). \tag{3.9}$$

Next it is easy to show that for $f \in C_{2\pi}$ we have

$$\lim_{n \to \infty} \lambda_{kn}(x) = 0, \qquad k = 2, 3, 4, 5. \tag{3.10}$$

Here we use (3.5)–(3.7) and follow as in Fejér sums. From (3.8) and (3.10) Theorem 1 follows at once.

ACKNOWLEDGMENTS

The authors are grateful to Professor Paul Nevai and the referee for their valuable suggestions.

REFERENCES

1. G. G. LORENTZ, K. JETTER, AND S. D. RIEMENSCHNEIDER, "Birkhoff Interpolation," Addison–Wesley, Reading, Mass., 1983.
2. J. MARCINKIEWICZ, Sur l'interpolation, I, *Studia Math.* **6** (1936), 1–17.
3. P. NEVAI, Mean convergence of Lagrange interpolation, I, II, *J. Approx. Theory* **18** (1976), 363–377; **30** (1980), 263–276.
4. P. NEVAI, Orthogonal polynomials, *Mem. Amer. Math. Soc.* **213** (1979).
5. P. NEVAI, Mean convergence of Lagrange interpolation, III, *Trans. Amer. Math. Soc.* **282** (2) (1984), 669–698.
6. A. SHARMA AND A. K. VARMA, Trigonometric interpolation (0, 2, 3) case, *Ann. Polon. Math.* **21** (1968), 51–58.
7. G. SZEGÖ, Orthogonal polynomials, *Amer. Math. Soc. Colloq. Publ.* **23** (1967).
8. A. K. VARMA, Concerning the order approximation of periodic continuous functions by trigonometric interpolation polynomials, II, *Ann. Polon. Math.* **28** (1973), 327–334.
9. P. VÉRTESI, Notes on mean convergence of Lagrange parabolas, *J. Approx. Theory* **28** (1980), 30–35.
10. A. ZYGMUND, "Trigonometric Series," Vols. I and II, Cambridge Univ. Press, London/New York, 1959.

Printed in Belgium